

# A Schatten -von Neumann class criterion for the magnetic Weyl calculus

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## Abstract

We prove criteria for a '*magnetic*' Weyl operator (see [15, 11]) to be in a Schatten-von Neuman class by extending a method developed by H. Cordes [5], T. Kato [9] and G. Arsou [1].

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## 1 Introduction

The '*magnetic*' Weyl quantization [15] is proven in [16] to be a *strict deformation quantization* in the sense of Rieffel [19, 20, 14] and its associated '*magnetic*' Weyl calculus is developed in [17, 11, 13] where a *magnetic version* of the Calderon-Vaillancourt Theorem is proven. In this paper we prove criteria for a '*magnetic*' Weyl operator to be in a Schatten-von Neuman class by extending a method developed by H. Cordes [5], T. Kato [9] and G. Arsou [1]. Such criteria may then be used in connection with the analysis of quantum Hamiltonians with magnetic fields. Our main result is formulated in Theorem 1.2.

Let us fix some general notations. Recall that for any  $a > 0$  we denote by  $[a] \in \mathbb{N}$  its integer part (i.e. the largest natural number less than or equal to  $a$ ). For any finite dimensional real vector space  $\mathcal{V}$ , we shall denote by  $BC(\mathcal{V})$  (resp.  $BC_u(\mathcal{V})$ ) the space of bounded continuous (resp. of bounded uniformly continuous) functions with the  $\|\cdot\|_\infty$  norm, by  $C^\infty(\mathcal{V})$  the space of smooth functions on  $\mathcal{V}$ , by  $C_{\text{pol}}^\infty(\mathcal{V})$  its subspace of smooth functions that are polynomially bounded together with all their derivatives and by  $BC^\infty(\mathcal{V})$  the subspace of smooth functions that are bounded together with all their derivatives; we consider all these spaces endowed with their usual locally convex topologies (see [21]). For any  $m \in \mathbb{R}$  and any Banach space  $\mathcal{B}$  we shall consider the function spaces  $S^m(\mathcal{V}; \mathcal{B})$  of  $\mathcal{B}$ -valued smooth functions on  $\mathcal{V}$  such that

$$\sup_{v \in \mathcal{V}} \langle v \rangle^{-m+M} \sum_{|\beta|=M} \|(\partial_v^\beta F)(v)\|_{\mathcal{B}} < \infty, \quad \forall M \in \mathbb{N}. \quad (1.1)$$

We shall denote by  $S^m(\mathcal{V}) := S^m(\mathcal{V}; \mathbb{C})$ . We shall consider the space of Schwartz test functions  $\mathcal{S}(\mathcal{V})$  endowed with its Fréchet topology and its dual  $\mathcal{S}'(\mathcal{V})$  and denote by  $\langle \cdot, \cdot \rangle_{\mathcal{V}}$  the associated duality map. We denote by  $\tau_v$  the translation with  $v \in \mathcal{V}$  (acting on the space of tempered distributions). We shall also consider the usual Sobolev spaces  $\mathcal{H}^m(\mathcal{V})$  of any order  $m \in \mathbb{R}$ , on  $\mathcal{V}$ . For any vector  $v \in \mathcal{V}$  we denote by  $\langle v \rangle := \sqrt{1 + |v|^2}$ . We denote the convolution operation by

$$(f * g)(v) := \int_{\mathcal{V}} f(v - u)g(u) du, \quad \forall (f, g) \in \mathcal{S}(\mathcal{V}) \times \mathcal{S}(\mathcal{V}) \quad (1.2)$$

and also its possible extensions to larger spaces of distributions on  $\mathcal{V}$ . For two linear topological spaces  $\mathcal{L}_1$  and  $\mathcal{L}_2$  we shall denote by  $\mathbb{B}(\mathcal{L}_1, \mathcal{L}_2)$  the linear space of continuous linear operators from  $\mathcal{L}_1$  to  $\mathcal{L}_2$ , endowed with the bounded convergence topology [4].

We shall work on the configuration space  $\mathcal{X} := \mathbb{R}^d$  and consider its dual  $\mathcal{X}^*$  with the duality map denoted by  $\langle \cdot, \cdot \rangle: \mathcal{X}^* \times \mathcal{X} \rightarrow \mathbb{R}$ . Let us also consider the *phase space*  $\Xi := \mathcal{X} \times \mathcal{X}^*$  with the canonical symplectic map  $\sigma(X, Y) := \langle \xi, y \rangle - \langle \eta, x \rangle$  for  $X := (x, \xi)$  and  $Y := (y, \eta)$  two arbitrary points of  $\Xi$ .

The *Weyl quantization* (see [6, 7, 8]) defines a linear topological isomorphism

$$\mathfrak{Op}: \mathcal{S}'(\Xi) \rightarrow \mathbb{B}(\mathcal{S}(\mathcal{X}); \mathcal{S}'(\mathcal{X})) \quad (1.3)$$

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for the strong topologies. Explicitly, for  $F \in \mathcal{S}(\mathcal{X})$  we have the formula

$$\mathfrak{Op}(F) = (2\pi)^{-d/2} \int_{\Xi} \left( (2\pi)^{-d/2} \int_{\Xi} e^{i\sigma(X,Y)} F(Y) dY \right) W(X) dX \equiv (2\pi)^{-d/2} \int_{\Xi} \mathcal{F}_{\Xi}^{-}[F](X) W(X) dX \quad (1.4)$$

$$(W((x, \xi))\phi)(z) := e^{(i/2)\langle \xi, x \rangle} e^{-i\langle \xi, z \rangle} \phi(z + x), \quad \forall \phi \in \mathcal{S}(\mathcal{X}). \quad (1.5)$$

We shall use some classes of Hörmander type symbols. For  $m \in \mathbb{R}$  and  $\rho \in [0, 1]$  let us define:

$$\nu_{N,M}^m(F) := \sup_{(x, \xi) \in \Xi} \langle \xi \rangle^{-m} \sum_{|\alpha|=N} \sum_{|\beta|=M} \left| (\partial_x^\alpha \partial_\xi^\beta F)(x, \xi) \right|, \quad \forall (N, M) \in \mathbb{N} \times \mathbb{N}, \quad \forall F \in C^\infty(\Xi), \quad (1.6)$$

$$S_\rho^m(\Xi) := \left\{ F \in C^\infty(\Xi) \mid \nu_{N,M}^{m-M\rho}(F) < \infty, \quad \forall (N, M) \in \mathbb{N} \times \mathbb{N} \right\}. \quad (1.7)$$

Evidently  $S^m(\mathcal{X}^*)$  may be considered as the subspace of  $S_1^m(\Xi)$  of functions constant in the directions in  $\mathcal{X}$ .

We shall usually work in the Hilbert space  $L^2(\mathcal{X})$  (defined with respect to the Lebesgue measure). In general for a complex Hilbert space  $\mathcal{K}$  we shall denote by  $(\cdot, \cdot)_{\mathcal{K}}$  its scalar product (supposed to be anti-linear in the first variable). For any Hilbert space  $\mathcal{K}$  we denote by  $\mathbb{B}(\mathcal{K})$  the  $C^*$ -algebra of bounded operators on  $\mathcal{K}$  and by  $\mathbb{B}_\infty(\mathcal{K})$  its ideal of compact operators.

**Definition 1.1.** Given a Hilbert space  $\mathcal{K}$ , for any  $p \in [1, \infty)$  we consider the linear subspace of compact operators  $A \in \mathbb{B}_\infty(\mathcal{K})$  with the property that

$$\exists \lim_{N \nearrow \infty} \sum_{n \leq N} \mu_n(A) < \infty, \quad (1.8)$$

where  $\{\mu_n(A)\}_{n \in \mathbb{N}}$  are the singular values of the operator  $A \in \mathbb{B}_\infty(\mathcal{K})$ , [3]. This subspace, denoted by  $\mathbb{B}_p(\mathcal{K})$  and called the Schatten-von Neumann class of order  $p$ , is a Banach space for the norm

$$\|A\|_{\mathbb{B}_p(\mathcal{K})} := \lim_{N \nearrow \infty} \left( \sum_{n \leq N} \mu_n(A)^p \right)^{1/p}. \quad (1.9)$$

We recall that  $\mathbb{B}_1(\mathcal{K})$  is the space of trace-class operators and  $\mathbb{B}_2(\mathcal{K})$  the space of Hilbert-Schmidt operators that is a Hilbert space for the scalar product  $(A, B)_{\mathbb{B}_2(\mathcal{K})} := \text{Tr}(A^* B)$ .

## 1.1 The magnetic Weyl calculus.

The magnetic fields are *closed 2-forms* on  $\mathcal{X}$  that we shall suppose to have components of class  $BC^\infty(\mathcal{X})$ . To any such magnetic field  $B$  one can associate in a highly non-unique way a vector potential  $A$ , i.e. a 1-form such that  $B = dA$ ; different choices for the vector potential are related by a change of gauge (i.e.  $dA = B = dA'$  if and only if  $\exists \varphi, A' = A + d\varphi$ ). We shall always suppose the vector potential to have components of class  $C_{\text{pol}}^\infty(\mathcal{X})$  because such a choice always exists for magnetic fields of class  $BC^\infty(\mathcal{X})$ . We use two important '*phase factors*' defined in terms of these exterior forms:

$$\Lambda^A(x, z) := \exp \left\{ -i \int_{[x, z]} A \right\} \quad (1.10)$$

$$\Omega^B(x, y, z) := \exp \left\{ -i \int_{\langle x, y, z \rangle} B \right\} \quad (1.11)$$

where  $[x, z]$  is the oriented line segment from  $x \in \mathcal{X}$  to  $z \in \mathcal{X}$  and  $\langle x, y, z \rangle$  is the oriented triangle of vertices  $\{x, y, z\} \subset \mathcal{X}$ . From Stoke's Theorem we deduce that  $\Omega^B(x, y, z) = \Lambda^A(x, y) \Lambda^A(y, z) \Lambda^A(z, x)$ .

Let us recall from [15] the *magnetic Weyl system* defined as the family of unitary operators in  $L^2(\mathcal{X})$ :

$$\{W^A(X)\}_{X \in \Xi}, \quad (W^A((x, \xi))u)(z) := \Lambda^A(z, z+x) (W((x, \xi))u)(z), \quad \forall u \in \mathcal{H}. \quad (1.12)$$

As explained in [15] they are defined as *unitary groups associated to the canonical observables in the minimal coupling formalism* for the vector potential  $A$ . With the help of this magnetic Weyl system one can define a *magnetic Weyl calculus* (i.e. a magnetic quantization) as in [15, 11]

$$\mathfrak{Op}^A(F) = (2\pi)^{-d/2} \int_{\Xi} \mathcal{F}_{\Xi}^{-}[F](X) W^A(X) dX. \quad (1.13)$$

Let us make the connection with the ‘*twisted integral kernels*’ formalism in [18]. For any integral kernel  $K \in \mathcal{S}'(\mathcal{X} \times \mathcal{X})$  one can associate its ‘magnetic’ twisted integral kernel

$$K^A(x, y) := \Lambda^A(x, y)K(x, y). \quad (1.14)$$

Let us denote by  $\mathcal{Int}K$  the corresponding linear operator on  $\mathcal{S}(\mathcal{X})$ ; i.e.  $(v, (\mathcal{Int}K)u)_{L^2(\mathcal{X})} = \langle K, \bar{v} \otimes u \rangle_{\mathcal{X}}$  for any  $(u, v) \in [\mathcal{S}(\mathcal{X})]^2$ . Let us recall the linear bijection  $\mathfrak{W} : \mathcal{S}'(\Xi) \rightarrow \mathcal{S}'(\mathcal{X} \times \mathcal{X})$  associated to the usual Weyl calculus (1.3) by the equality  $\mathfrak{Op}(F) = \mathcal{Int}(\mathfrak{W}F)$ :

$$(\mathfrak{W}F)(x, y) := (2\pi)^{-d} \int_{\mathcal{X}^*} e^{i\langle \xi, x-y \rangle} F\left(\frac{x+y}{2}, \xi\right) d\xi. \quad (1.15)$$

Then we have the equality

$$\mathfrak{Op}^A(F) = \mathcal{Int}(\Lambda^A \mathfrak{W}F). \quad (1.16)$$

This functional calculus induces a *magnetic Moyal product*  $\sharp^B : \mathcal{S}(\Xi) \times \mathcal{S}(\Xi) \rightarrow \mathcal{S}(\Xi)$  such that  $\mathfrak{Op}^A(f \sharp^B g) = \mathfrak{Op}^A(f) \mathfrak{Op}^A(g)$ . Explicitely we have

$$(f \sharp^B g) = \pi^{-2d} \int_{\Xi} \int_{\Xi} e^{-2i\sigma(Y, Z)} \Omega^B(x - y - z, x + y - z, x - y + z) f(X - Y) g(X - Z) dY dZ \quad (1.17)$$

as oscillating integrals (see [8]). We shall use the notation

$$\omega^B(x, y, z) := \Omega^B(x - y - z, x + y - z, x - y + z). \quad (1.18)$$

In [11] one gives the extension of this magnetic Moyal product to the usual Hörmander type symbols and in [11, 13] it is proven that this calculus has similar properties with the usual Moyal product. If a symbol  $F \in \mathcal{S}'(\Xi)$  is invertible for this *magnetic Moyal product* we shall denote by  $F_B^-$  its inverse.

In [11] it is proven that for any symbol  $F \in S_0^0(\mathcal{X})$  the operator norm of  $\mathfrak{Op}^A(F)$  is bounded by some seminorm defining the Fréchet topology on  $S_0^0(\mathcal{X})$  and this seminorm only depends on the dimension  $d$  of  $\mathcal{X}$  and some Fréchet seminorm of the components of the magnetic field in  $BC^\infty(\mathcal{X})$  (this second fact, although not explicitly stated there, easily follows when looking at the detailed proof of Theorem 3.1 in [12]). We shall define the following associated norm on the  $S_0^0(\mathcal{X})$  symbols:

$$\|F\|_B := \|\mathfrak{Op}^A(F)\|_{\mathbb{B}(\mathcal{H})}. \quad (1.19)$$

In [15] it is proven that  $\mathfrak{Op}^A(F)$  is Hilbert-Schmidt if and only if  $F \in L^2(\Xi)$  and  $\|F\|_{L^2(\Xi)} = \|\mathfrak{Op}^A(F)\|_{\mathbb{B}_2(\mathcal{H})}$ .

## 1.2 The main result.

In the papers [1, 2], G. Arsu uses some ideas and results of H.O. Cordes [5] and T. Kato [9] and the characterization of Schatten-von Neumann classes of operators coming from J.W. Calkin and R. Schatten (see [3, 23]) in order to obtain an interesting criterion for a Weyl operator to be in a given Schatten-von Neumann class. Our aim in this paper is to replace the usual Weyl system with the magnetic Weyl system (1.12) and prove a criterion for a *magnetic Weyl operator* (1.13) to be in a given Schatten-von Neumann class. We prove the following Theorem.

**Theorem 1.2.** *Suppose that  $B$  is a magnetic field with components of class  $BC^\infty(\mathcal{X})$  and we choose some vector potential  $A$  for  $B$  with components of class  $C_{\text{pol}}^\infty(\mathcal{X})$ . Suppose that  $F \in \mathcal{S}'(\Xi)$  and let us denote by  $s(d) := 2[d/2] + 2$  and  $t(d) := d + [d/2] + 1$ .*

1. *If  $\partial_x^\alpha \partial_\xi^\beta F \in L^\infty(\Xi)$  for  $|\alpha| \leq s(d)$  and  $|\beta| \leq t(d)$ , then  $\mathfrak{Op}^A(F) \in \mathbb{B}(L^2(\mathcal{X}))$  and there exists some finite constant  $C > 0$  such that*

$$\|\mathfrak{Op}^A(F)\|_{\mathbb{B}(\mathcal{H})} \leq C \sum_{|\alpha| \leq s(d)} \sum_{|\beta| \leq t(d)} \left\| \partial_x^\alpha \partial_\xi^\beta F \right\|_{L^\infty(\Xi)}.$$

2. *For  $p \in [1, \infty)$  if  $\partial_x^\alpha \partial_\xi^\beta F \in L^p(\Xi)$  for  $|\alpha| \leq s(d)$  and  $|\beta| \leq t(d)$ , then  $\mathfrak{Op}^A(F) \in \mathbb{B}_p(L^2(\mathcal{X}))$  and there exists some finite constant  $\tilde{C}_p > 0$  such that*

$$\|\mathfrak{Op}^A(F)\|_{\mathbb{B}_p(\mathcal{H})} \leq \tilde{C}_p \sum_{|\alpha| \leq s(d)} \sum_{|\beta| \leq t(d)} \left\| \partial_x^\alpha \partial_\xi^\beta F \right\|_{L^p(\Xi)}.$$

3. If  $\partial_x^\alpha \partial_\xi^\beta F \in L^\infty(\Xi)$  for  $|\alpha| \leq s(d)$  and  $|\beta| \leq t(d)$ , and  $\lim_{(x,\xi) \rightarrow \infty} (\partial_x^\alpha \partial_\xi^\beta F)(x, \xi) = 0$  for  $|\alpha| \leq s(d)$  and  $|\beta| \leq t(d)$ , then  $\mathfrak{Op}^A(F) \in \mathbb{B}_\infty(L^2(\mathcal{X}))$ .

**Remark 1.3.** We note that points (1) and (2) of the Theorem are the '*magnetic*' version of Theorem 6.4 in [1]. First, let us consider the value of  $s(d) \in \mathbb{N}$  (the number of derivatives with respect to the  $\mathcal{X}$ -variables) that we obtain. For  $d \in \mathbb{N}$  odd, we have  $s(d) = d + 1$  exactly as in [1], while for  $d \in \mathbb{N}$  even we have  $s(d) = d + 2$  that is larger by one unit with respect to the value in [1]; this is just the consequence of our choice to work without fractionary derivatives, that would largely complicate the technical arguments without a real improvement. Concerning  $t(d) \in \mathbb{N}$ , it is interesting to note that it is larger then its value given in [1] for the zero magnetic field situation and that reflects the fact that the presence of a magnetic field that does not vanish at infinity obliges us to control several derivatives of the symbol. Moreover, if we go into the details of our proof of Theorem 1.2 (more precisely the proof of Proposition 2.8) we easily see that in the absence of the magnetic field (i.e. of the factor  $\omega^{\tau_z B}$ ) we can take  $t(d) = d + 1$  as in [1]. Let us also note that points (1) and (3) in our Theorem are similar to Theorem 1 in [10] but with assumptions on fewer derivatives of the symbol.

## 2 Proof of Theorem 1.2.

While the idea of the proof follows closely the arguments and some results from [1, 5, 9], several essential technical steps have to be completely reconsidered in order to control the '*magnetic phase factors*' present in the magnetic Weyl calculus.

Let us recall that in [1, 9] one begins by noticing that the fundamental solutions of some simple elliptic differential operators are symbols of trace-class operators (as implied by Cordes Lemma [5, 2]) and starting from the following formula, valid for two symbols  $f$  and  $g$  of class  $\mathcal{S}(\Xi)$ ,

$$\mathfrak{Op}(f * g) = \int_{\Xi} f(X) \mathfrak{Op}(\tau_{-X} g) dX = \int_{\Xi} f(X) \left( W(-X) \mathfrak{Op}(g) W(X) \right) dX, \quad (2.20)$$

a procedure elaborated by G. Arsu [1] using the results of J.W. Calkin and R. Schatten (see [3, 22, 23]) and some ideas of T. Kato [9] allows to obtain the desired result. Let us develop these ideas and adapt them to our situation.

For  $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$  let us consider the following  $\Psi$ DO on  $\Xi$ :

$$\mathfrak{L}_{s,t} := (\mathbf{1} - \Delta_{\mathcal{X}})^{s/2} (\mathbf{1} - \Delta_{\mathcal{X}^*})^{t/2} \quad (2.21)$$

where

$$\Delta_{\mathcal{X}} := \sum_{1 \leq j \leq d} \partial_{x_j}^2, \quad \Delta_{\mathcal{X}^*} := \sum_{1 \leq j \leq d} \partial_{\xi_j}^2. \quad (2.22)$$

Let us denote by  $\psi_s \in \mathcal{S}'(\mathcal{X})$  the unique fundamental solution of  $(\mathbf{1} - \Delta_{\mathcal{X}})^{s/2}$  and by  $\dot{\psi}_t \in \mathcal{S}'(\mathcal{X}^*)$  the unique fundamental solution of  $(\mathbf{1} - \Delta_{\mathcal{X}^*})^{t/2}$ . Let us recall the following well known result (see for example section 5 in [1] and Corollary 2.6 in [2] for the last statement).

**Proposition 2.4.** For any  $s > 0$  the distribution  $\psi_s \in \mathcal{S}'(\mathcal{X})$  is in fact a function of class  $L^1(\mathcal{X})$  that is in  $\mathcal{S}(\mathcal{X} \setminus \{0\})$ . For  $|x| \searrow 0$  we have that

$$\partial_x^\alpha \psi_s \sim \mathcal{O}(1 + |x|^{s-d-|\alpha|}), \quad s - d - |\alpha| \neq 0, \quad (2.23)$$

$$\partial_x^\alpha \psi_s \sim \mathcal{O}(1 + \ln|x|^{-1}), \quad s - d - |\alpha| = 0. \quad (2.24)$$

For  $s > d$  we have that  $\psi_s \in \mathcal{H}^p(\mathcal{X})$  for any  $p < (s/2)$ . We have evidently a similar behaviour for  $\dot{\psi}_t \in \mathcal{S}'(\mathcal{X}^*)$ .

This result and the Cordes Lemma [5, 2]) allow to prove that  $\psi_s \otimes \dot{\psi}_t$  is an integral kernel defining a trace-class operator. Then, using 2.20 and the trivial fact that for any  $f \in \mathcal{S}'(\Xi)$ , if we denote by  $\delta_0$  the Dirac measure of mass 1 at  $0 \in \Xi$  we have that

$$f = f * \delta_0 = f * (\mathfrak{L}_{s,t}(\psi_s \otimes \dot{\psi}_t)) = (\mathfrak{L}_{s,t} f) * (\psi_s \otimes \dot{\psi}_t)$$

and *Kato's operator calculus* in [9] and Lemma 4.3 in [1] give the desired result in the absence of the magnetic field. An important difficulty for the case of the '*magnetic*' Weyl calculus comes from the fact that equation (2.20) is no longer valid for the magnetic Weyl calculus; more precisely we have

$$\mathfrak{Op}^A(\tau_{-X} g) \neq W^A(X)^* \mathfrak{Op}^A(g) W^A(X). \quad (2.25)$$

The following subsection is devoted to the control of this difficulty.

## 2.1 Magnetic translations of symbols.

In Proposition 3.4 in [13] one defines the action of  $\Xi$  on the symbols in  $\mathcal{S}'(\Xi)$  by '*magnetic translations*':

$$\Xi \ni Z \mapsto \mathfrak{T}_Z^B \in \mathbb{B}(\mathcal{S}'(\Xi); \mathcal{S}'(\Xi)) \quad (2.26)$$

as the conjugate action associated to the *magnetic Weyl system*:

$$\mathfrak{Op}^A(\mathfrak{T}_{-Z}^B g) := W^A(Z)^* \mathfrak{Op}^A(g) W^A(Z). \quad (2.27)$$

Explicitly we have the formula:

$$\mathfrak{T}_Z^B g = [(\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) e^{-iS_z^B}] \star [\tau_Z g], \quad \forall g \in \mathcal{S}'(\Xi) \quad (2.28)$$

where we have used (as in [13]) the *inverse Fourier transform* on  $\mathcal{X}$

$$(\mathcal{F}_{\mathcal{X}}^- \phi)(\xi) := (2\pi)^{-d/2} \int_{\mathcal{X}} e^{i\langle \xi, y \rangle} \phi(y) dy, \quad \forall \phi \in \mathcal{S}(\mathcal{X}), \quad (2.29)$$

the notation

$$S_z^B(x, y) := - \sum_{j \neq k} y_j z_k \int_{-1/2}^{1/2} ds \int_0^1 dt B_{jk}(x + sy + tz), \quad (2.30)$$

and the following '*mixed*' product:

$$(f \star g)(x, \xi) := \int_{\mathcal{X}^*} f(x, \eta) g(x, \xi - \eta) d\eta. \quad (2.31)$$

We note that  $S_z^B(x, y)$  as defined in (2.30) is in fact the flux of the magnetic field  $B$  through the oriented parallelogram of vertices  $\{x + (y/2), x - (y/2), x - (y/2) + z, x + (y/2) + z\}$ . We shall also use the notation

$$\Theta_z^B := [(\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) e^{-iS_z^B}] \in \mathcal{S}'(\Xi) \quad (2.32)$$

and remark that  $\Theta_z^B \star \Theta_z^{-B} = 1 \otimes \delta_0$ , the identity element for the '*mixed*' product  $\star$ . We also remark that

$$\mathfrak{T}_Z^B f := \Theta_z^B \star [\tau_Z g] = \tau_Z [\Theta_z^{\tau_z B} \star f]. \quad (2.33)$$

These arguments allow us to write

$$\begin{aligned} \mathfrak{Op}^A(f \star g) &= \int_{\Xi} f(Z) \mathfrak{Op}^A(\tau_{-Z} g) dZ = \int_{\Xi} f(Z) \mathfrak{Op}^A(\tau_{-Z} (\Theta_{-Z}^{\tau_z B} \star \Theta_{-Z}^{-\tau_z B} \star g)) dZ = \\ &= \int_{\Xi} f(Z) \mathfrak{Op}^A(\mathfrak{T}_{-Z}^B (\Theta_{-Z}^{-\tau_z B} \star g)) dZ = \int_{\Xi} f(Z) W^A(Z)^* \mathfrak{Op}^A(\Theta_{-Z}^{-\tau_z B} \star g) W^A(Z) dZ. \end{aligned} \quad (2.34)$$

This last formula replaces (2.20) in the case of the '*magnetic*' Weyl calculus.

## 2.2 Kato's operator calculus.

We recall here one of the main results in [1] using the *operator calculus* elaborated by T. Kato in [9]. Suppose given a measurable map  $W : \Xi \rightarrow \mathbb{B}(\mathcal{H})$  for the weak operator topology on  $\mathbb{B}(\mathcal{H})$ . For any trace-class operator  $T \in \mathbb{B}_1(\mathcal{H})$  and any  $\varphi \in \mathcal{S}(\Xi)$  we can define the following integral (with respect to the weak operator topology):

$$\varphi\{T\} := \int_{\Xi} \varphi(X) (W(X)^* T W(X)) dX. \quad (2.35)$$

**Proposition 2.5.** (Lemma 4.3 in [1])

1. If there exists a finite  $C > 0$  such that

$$\int_{\Xi} |(u, W(X)v)_{\mathcal{H}}|^2 dX \leq C \|u\|_{\mathcal{H}}^2 \|v\|_{\mathcal{H}}^2, \quad \forall (u, v) \in \mathcal{H} \times \mathcal{H}, \quad (2.36)$$

then for any  $\varphi \in L^\infty(\Xi)$  the integral (2.35) is well defined in the weak operator topology on  $\mathbb{B}(\mathcal{H})$  and we have the estimation

$$\|\varphi\{T\}\|_{\mathbb{B}(\mathcal{H})} \leq C \|\varphi\|_{L^\infty(X)} \|T\|_{\mathbb{B}_1(\mathcal{H})}. \quad (2.37)$$

2. If there exists a finite  $C > 0$  such that  $\|W(X)\|_{\mathbb{B}(\mathcal{H})} \leq \sqrt{C}$  almost everywhere on  $\Xi$ , then for any  $\varphi \in L^1(\Xi)$  the integral (2.35) is well defined in the weak operator topology on  $\mathbb{B}(\mathcal{H})$ , belongs to  $\mathbb{B}_1(\mathcal{H})$  and we have the estimation

$$\|\varphi\{T\}\|_{\mathbb{B}_1(\mathcal{H})} \leq C\|\varphi\|_{L^1(X)}\|T\|_{\mathbb{B}_1(\mathcal{H})}. \quad (2.38)$$

3. If the map  $W : \Xi \rightarrow \mathbb{B}(\mathcal{H})$  satisfies both conditions above for some finite  $C > 0$ , then for any  $\varphi \in L^p(\Xi)$ , for some  $p \in (1, \infty)$ , the integral (2.35) is well defined in the weak operator topology on  $\mathbb{B}(\mathcal{H})$ , belongs to  $\mathbb{B}_p(\mathcal{H})$  and we have the estimation

$$\|\varphi\{T\}\|_{\mathbb{B}_p(\mathcal{H})} \leq C\|\varphi\|_{L^p(X)}\|T\|_{\mathbb{B}_1(\mathcal{H})}. \quad (2.39)$$

**Remark 2.6.** We note that for any vector potential, our magnetic Weyl system  $\Xi \ni X \mapsto W^A(X) \in \mathbb{B}(\mathcal{H})$  satisfies both conditions in Theorem 2.5 with a constant  $C = 1$ , the first one as proven in Proposition 3.8 (a) in [15] and the second one due to their unitarity.

Using (2.34) and the above Remark we obtain the following Corollary of Proposition 2.5 (the ‘magnetic version’ of Theorem 4.5 in [1]):

**Corollary 2.7.** Suppose given a magnetic field  $B$  with components of class  $BC^\infty(\mathcal{X})$  and suppose fixed some vector potential  $A$  for  $B$  with components of class  $C_{\text{pol}}^\infty(\mathcal{X})$ ; if a symbol  $F \in \mathcal{S}'(\Xi)$  has the property  $\mathfrak{Op}^A(F) \in \mathbb{B}_1(\mathcal{H})$ , then

1. For any  $f \in L^\infty(\Xi)$  we have that  $\mathfrak{Op}^A(f * F) \in \mathbb{B}(\mathcal{H})$  and  $\|\mathfrak{Op}^A(f * F)\|_{\mathbb{B}(\mathcal{H})} \leq \|f\|_{L^\infty(X)}\|\mathfrak{Op}^A(F)\|_{\mathbb{B}_1(\mathcal{H})}$ .
2. For any  $f \in L^p(\Xi)$ , for some  $p \in [1, \infty)$ , we have that  $\mathfrak{Op}^A(f * F) \in \mathbb{B}_p(\mathcal{H})$  and

$$\|\mathfrak{Op}^A(f * F)\|_{\mathbb{B}_p(\mathcal{H})} \leq \|f\|_{L^p(X)}\|\mathfrak{Op}^A(F)\|_{\mathbb{B}_1(\mathcal{H})}.$$

## 2.3 Estimations for $\mathfrak{Op}^A(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t})$ .

Thus in order to finish the proof of our Theorem 1.2, we only have to prove that  $\mathfrak{Op}^A(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t}) \in \mathbb{B}_1(\mathcal{H})$  for  $\Psi_{s,t} := \psi_s \otimes \dot{\psi}_t$  with  $s > 0$  and  $t > 0$  large enough (with the notations introduced at the beginning of Section 2).

We use (1.16) in connection with (2.32) and (2.30), denote by  $\tilde{\Psi}_{s,t} := \mathfrak{W}\Psi_{s,t} \in \mathcal{S}'(\mathcal{X} \times \mathcal{X})$  and write

$$\mathfrak{Op}^A(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t}) = \text{Int}(\Lambda^A \mathfrak{W}(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t})), \quad (2.40)$$

$$\begin{aligned} (\mathfrak{W}(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t}))(x, y) &= \exp \left\{ -iS_{-z}^{-\tau_z B}((x+y)/2, y-x) \right\} \tilde{\Psi}_{s,t}(x, y) \\ &= \exp \left\{ iS_{-z}^{\tau_z B}((x+y)/2, y-x) \right\} \tilde{\Psi}_{s,t}(x, y), \end{aligned} \quad (2.41)$$

$$S_{-z}^{\tau_z B}((x+y)/2, y-x) = -\sum_{j \neq k} (y_j - x_j) z_k \int_{-1/2}^{1/2} ds \int_0^1 dt B_{jk}((x+y)/2 + s(y-x) + tz), \quad (2.42)$$

and this is the flux of the magnetic field  $B$  through the oriented parallelogram of vertices  $\{y, x, x+z, y+z\}$ . Using the relation  $B = dA$  and Stoke’s Theorem we conclude that

$$\begin{aligned} \Lambda^A(x, y) \exp \left\{ iS_{-z}^{\tau_z B}((x+y)/2, y-x) \right\} &= \Lambda^A(x, y) \Lambda^A(y, x) \Lambda^A(x, x+z) \Lambda^A(x+z, y+z) \Lambda^A(y+z, y) \\ &= \Lambda^A(x, x+z) \Lambda^A(x+z, y+z) \Lambda^A(y+z, y) \\ &= \Lambda^A(x, x+z) \Lambda^{\tau_z A}(x, y) \Lambda^A(y+z, y). \end{aligned} \quad (2.43)$$

Thus, if we denote by  $\mathfrak{U}_z^A$  the unitary operator in  $L^2(\mathcal{X})$  defined by multiplication with the function  $x \mapsto \Lambda^A(\cdot, \cdot + z)$ , we can write that  $\mathfrak{Op}^A(\Theta_{-z}^{-\tau_z B} \star \Psi_{s,t}) = \mathfrak{U}_z^A \mathfrak{Op}^{\tau_z A}(\Psi_{s,t}) [\mathfrak{U}_z^A]^{-1}$  and it is enough to prove the following ‘magnetic version’ of Lemma 1 in [5].

**Proposition 2.8.** Suppose given a magnetic field  $B = dA$  with components of class  $BC^\infty(\mathcal{X})$ ; for  $t > 3d/2$  and  $s > 2[d/2] + 2$  we have that  $\mathfrak{Op}^{\tau_z A}(\Psi_{s,t}) \in \mathbb{B}_1(\mathcal{H})$  uniformly for  $z \in \mathcal{X}$ .

*Proof.* We shall proceed as in [5, 2] but we shall work with the magnetic Moyal product (1.17). The idea is to write  $\Psi_{s,t}$  as a magnetic Moyal product of two symbols of class  $L^2(\Xi)$ :

$$\Psi_{s,t} = \Phi^{(1)} \sharp^{\tau_z B} \Phi^{(2)}, \quad \Phi^{(j)} \in L^2(\Xi), \quad j = 1, 2. \quad (2.44)$$

Let us consider the symbols  $p_{m,\lambda}(X) := \langle \xi \rangle^m + \lambda$  for any  $m > 0$  and some  $\lambda > 0$  large enough; they are evidently elliptic symbols of class  $S^m(\mathcal{X}^*)$  that, for  $\lambda > 0$  large enough, are invertible for the *magnetic Moyal product* due to Theorem 1.8 in [17]. More precisely, looking at the proof of this cited Theorem we see that

$$r_{m,\lambda} := (p_{m,\lambda})_{\tau_z B}^- = (\langle \xi \rangle^m + \lambda)^{-1} \sharp^{\tau_z B} \left( \sum_{k \in \mathbb{N}} \underbrace{s_{z,m}(\lambda) \sharp^{\tau_z B} \dots \sharp^{\tau_z B} s_{z,m}(\lambda)}_k \right) \quad (2.45)$$

with  $s_{z,m}(\lambda) \in S^{-\kappa}(\mathcal{X})$  for some  $\kappa \in (0, 1)$ , having the operator norm strictly less than 1 for  $\lambda > 0$  large enough and the defining Fréchet seminorms bounded by some seminorm of the components of  $\tau_z B$  in  $BC^\infty(\mathcal{X})$ ; as these seminorms are translation invariant, we have uniform bounds for  $z \in \mathcal{X}$ . Thus, using Proposition 3.10 in the Appendix and Proposition 6.2 in [13] we conclude that for  $\lambda > 0$  large enough, the symbol seminorms of  $(p_{m,\lambda})_{\tau_z B}^- \in S_1^{-m}(\mathcal{X})$  are bounded by some constants that do not depend on  $z \in \mathcal{X}$ .

Let us also consider the function  $q_r(X) := \langle x \rangle^r$  with  $r \in \mathbb{R}$ , defining a symbol of class  $S^r(\mathcal{X})$  for any  $r \in \mathbb{R}$  and also of class  $S_1^0(\Xi)$  for any  $r \leq 0$ . Formally we can write

$$\Psi_{s,t} = (q_{-r} \sharp^{\tau_z B} r_{m,\lambda}) \sharp^{\tau_z B} (p_{m,\lambda} \sharp^{\tau_z B} q_r \sharp^{\tau_z B} \Psi_{s,t}) \quad (2.46)$$

Using once again Proposition 3.10 in the Appendix and the fact that the seminorms of the components of the magnetic field that control the magnetic Moyal products are translation invariant, we easily conclude that for  $r > 0$ ,  $m > 0$  and  $\lambda > 0$  large enough

$$(q_{-r} \sharp^{\tau_z B} r_{m,\lambda}) \in S_1^{-m}(\mathcal{X}), \quad (2.47)$$

uniformly for  $z \in \mathcal{X}$ . Moreover, for any  $a \geq 0$  and  $b \geq 0$  we can write:

$$\begin{aligned} & \langle x \rangle^a \langle \xi \rangle^b (q_{-r} \sharp^{\tau_z B} r_{m,\lambda})(x, \xi) = \\ &= \pi^{-2d} \langle x \rangle^a \langle \xi \rangle^b \int_{\Xi \times \Xi} e^{-2i\sigma(Y, Y')} \omega^{\tau_z B}(x, y, y') \langle x - y \rangle^{-r} r_{m,\lambda}(x - y', \xi - \eta') dY dY' = \\ &= \pi^{-d} C_a \int_{\mathcal{X}} \langle x - y \rangle^{-(r-a)} \left( \frac{\langle x \rangle^a}{\langle x - y \rangle^a \langle y \rangle^a} \right) \times \\ & \times \left[ \int_{\mathcal{X}^*} \langle y \rangle^a \langle \eta' \rangle^b e^{2i\langle \eta', y \rangle} \left( \frac{\langle \xi \rangle^b}{\langle \xi - \eta' \rangle^b \langle \eta' \rangle^b} \right) (\langle \xi - \eta' \rangle^b r_{m,\lambda}(x, \xi - \eta')) d\eta' \right] dy. \end{aligned} \quad (2.48)$$

We use the identities:

$$\langle y \rangle^{2N_1} e^{2i\langle \eta', y \rangle} = (1 - 4^{-1} \Delta_{\eta'})^{N_1} e^{2i\langle \eta', y \rangle}, \quad \langle \eta' \rangle^{2N_2} e^{2i\langle \eta', y \rangle} = (1 - 4^{-1} \Delta_y)^{N_2} e^{2i\langle \eta', y \rangle} \quad (2.49)$$

and after some integrations by parts as in the proof of Proposition 3.10 in the Appendix, taking  $0 \leq a \leq r$ ,  $0 \leq b \leq m$  and  $2N_1 \geq [a] + d + 1$ ,  $2N_2 \geq [b] + d + 1$  we get that

$$\langle x \rangle^a \langle \xi \rangle^b |(q_{-r} \sharp^{\tau_z B} r_{m,\lambda})(x, \xi)| \leq C_{a,d} \sup_{(x,\xi) \in \Xi} \langle \xi \rangle^b \sum_{|\alpha| \leq 2N_2} |(\partial_\xi^\alpha r_{m,\lambda})(x, \xi)| \leq C(a, b) \nu_{0,2N_2}^m(r_{m,\lambda}). \quad (2.50)$$

A similar computation can be made for any derivative  $\partial_x^\alpha \partial_\xi^\beta (q_{-r} \sharp^{\tau_z B} r_{m,\lambda})$  so that we conclude that

$$q_a p_{b,0} (q_{-r} \sharp^{\tau_z B} r_{m,\lambda}) \in S_1^0(\mathcal{X}), \quad \forall (a, b) \in [0, r] \times [0, m] \quad (2.51)$$

and taking  $r > d/2$  and  $m > d/2$  we note that  $\Phi^{(1)} := q_{-r} \sharp^{\tau_z B} r_{m,\lambda} \in L^2(\Xi)$  so that  $\mathfrak{Op}^A(\Phi^{(1)}) \in \mathbb{B}_2(L^2(\mathcal{X}))$ .

Now let us study the second factor in (2.46). We note that for  $m > 0$  and  $r > 0$  the first two functions of this second magnetic Moyal product, namely  $p_{m,\lambda}$  and  $q_r$ , are in fact  $C^\infty(\Xi)$  functions with polynomial growth at infinity uniformly for all their derivatives, and thus Proposition 4.23 in [15] shows that their magnetic Moyal product may be well defined in the sense of tempered distributions and moreover this product (as a tempered distribution) may be further composed by magnetic Moyal product with any tempered distribution on  $\Xi$ . Thus  $\Phi^{(2)}$  is well defined as a tempered distribution on  $\Xi$  and we can also use the associativity of the magnetic Moyal product. Let us note that this tempered distribution depends in fact on  $z \in \mathcal{X}$  due to the translated magnetic field appearing in the two 'magnetic' Moyal products in the definition of  $\Phi^{(2)}$  as the second paranthesis in (2.46); thus we shall use the notation  $\Phi_z^{(2)}$  and notice that this dependence is uniformly smooth with respect to the weak distribution topology.

We begin by computing  $q_r \#^{\tau_z B} \Psi_{s,t} = q_r \#^{\tau_z B} (\psi_s \otimes \dot{\psi}_t)$  for  $r > d/2 > 0$ :

$$\begin{aligned} & [q_r \#^{\tau_z B} (\psi_s \otimes \dot{\psi}_t)](x, \xi) = \\ &= \pi^{-2d} \int_{\Xi \times \Xi} e^{-2i\sigma(Y, Y')} \omega^{\tau_z B}(x, y, y') < x - y >^r \psi_s(x - y') \dot{\psi}_t(\xi - \eta') dY dY' = \\ &= \pi^{-d} \psi_s(x) \int_{\mathcal{X} \times \mathcal{X}^*} e^{2i\langle \eta', y \rangle} < x - y >^r \dot{\psi}_t(\xi - \eta') dy d\eta' = \end{aligned} \quad (2.52)$$

$$= \pi^{-d} \psi_s(x) \int_{\mathcal{X}} e^{2i\langle \xi, y \rangle} < x - y >^r \left( \int_{\mathcal{X}^*} e^{-2i\langle \xi - \eta', y \rangle} \dot{\psi}_t(\xi - \eta') d\eta' \right) dy = \quad (2.53)$$

$$= 2^d \psi_s(x) \int_{\mathcal{X}} e^{2i\langle \xi, y \rangle} \frac{< x - y >^r}{< 2y >^t} dy = 2^d (q_r \psi_s)(x) \int_{\mathcal{X}} e^{2i\langle \xi, y \rangle} \frac{< x - y >^r}{< x >^r < 2y >^t} dy = \quad (2.54)$$

$$= (2\pi)^{d/2} (q_r \psi_s)(x) ((\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) f)(x, \xi) \quad (2.55)$$

where:

$$f(x, y) := \frac{< x - (y/2) >^r}{< x >^r < y >^t}. \quad (2.56)$$

It is easy to check that  $f \in C_{\text{pol}}^\infty(\mathcal{X} \times \mathcal{X})$  and satisfies the estimations:

$$|(\partial_x^\alpha \partial_y^\beta f)(x, y)| \leq C_{\alpha\beta} < x >^{-|\alpha|} < y >^{r-t-|\beta|}. \quad (2.57)$$

Now let us consider some  $m > d/2$ , and use the notations:  $\tilde{f} := (2\pi)^{d/2} (\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) f$  and for any  $r \geq 0$  the function  $\tilde{\psi}_{s,r}(x) := < x >^r \psi_s(x)$ . We notice that for any  $r \geq 0$  the function  $\tilde{\psi}_{s,r}$  has exactly the same properties as those of  $\psi_s$  given in Proposition 2.4.

We want to show that:

$$\Phi_z^{(2)} := p_{m,\lambda} \#^{\tau_z B} q_r \#^{\tau_z B} \Psi_{s,t} = (2\pi)^{d/2} \left( p_{m,\lambda} \#^{\tau_z B} [(q_r \psi_s \otimes 1) ((\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) f)] \right) \quad (2.58)$$

as a tempered distribution on  $\Xi$  is in fact an  $L^2(\Xi)$  function uniformly for  $z \in \mathcal{X}$ . In order to deal with the possible singularities of this distribution we shall regularize it by introducing 4 cut-off functions in the oscillatory integrals appearing in the definition (1.17), more precisely we shall approach  $\Phi_z^{(2)}$ , in the weak distribution topology, by the following continuous functions on  $\Xi$  depending also on 4 positive parameters  $\{R_j\}_{j=1,2,3,4}$ :

$$\begin{aligned} \widetilde{\Phi^{(2)}}_{(R_j, z)}(x, \xi) &:= \int_{\Xi} \int_{\Xi} e^{-2i\langle \eta, y' \rangle} e^{2i\langle \eta', y \rangle} \chi_{R_1}(y) \chi_{R_2}(y') \chi_{R_3}(\eta) \chi_{R_4}(\eta') \times \\ &\times p_{m,\lambda}(\xi - \eta) \tilde{\psi}_{s,r}(x - y') \tilde{f}(x - y', \xi - \eta') \omega^{\tau_z B}(x, y, y') dy dy' d\eta d\eta', \end{aligned} \quad (2.59)$$

where for any  $R > 0$  we define  $\chi_R(v) := \chi(R^{-1}|v|)$  with  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a smooth decreasing function that satisfies  $\chi(t) = 1$  for  $0 \leq t \leq 1$  and  $\chi(t) = 0$  for  $t \geq 2$ .

We shall first consider the term  $< \xi - \eta >^m$  in the function  $p_{m,\lambda}(\xi - \eta) = < \xi - \eta >^m + \lambda$  and the associated integral

$$\begin{aligned} \widetilde{\Phi^{(3)}}_{(R_j, z)}(x, \xi) &:= \int_{\Xi} \int_{\Xi} e^{-2i\langle \eta, y' \rangle} e^{2i\langle \eta', y \rangle} \chi_{R_1}(y) \chi_{R_2}(y') \chi_{R_3}(\eta) \chi_{R_4}(\eta') \times \\ &\times < \xi - \eta >^m \tilde{\psi}_{s,r}(x - y') \tilde{f}(x - y', \xi - \eta') \omega^{\tau_z B}(x, y, y') dy dy' d\eta d\eta'. \end{aligned} \quad (2.60)$$

We make the measure preserving change of variables:

$$(y, y', \eta, \eta') \mapsto (y, u, \zeta, \zeta'); \quad \begin{cases} u := x - y' \\ \zeta := \xi - \eta \\ \zeta' := \xi - \eta', \end{cases} \quad (2.61)$$

so that (2.60) may be written as

$$\widetilde{\Phi^{(3)}}_{(R_j, z)}(x, \xi) := \int_{\Xi} \int_{\Xi} e^{-2i\langle \xi, x - u - y \rangle} e^{2i\langle \zeta, x - u \rangle} e^{-2i\langle \zeta', y \rangle} \chi_{R_1}(y) \chi_{R_2}(x - u) \chi_{R_3}(\xi - \zeta) \chi_{R_4}(\xi - \zeta') \times \quad (2.62)$$



$$\begin{aligned} & \times \langle \zeta \rangle^m \tilde{\psi}_{s,r}(u) \tilde{f}(u, \zeta') \omega^{\tau_z B}(x, y, x-u) dy du d\zeta d\zeta' = \\ & = \int_{\mathcal{X}^*} e^{2i\langle \zeta, x \rangle} \langle \zeta \rangle^m \chi_{R_3}(\xi - \zeta) \left\{ \int_{\mathcal{X}} e^{-2i\langle \zeta, u \rangle} \tilde{\psi}_{s,r}(u) \chi_{R_2}(x-u) \times \right. \end{aligned} \quad (2.63)$$

$$\begin{aligned} & \times \left[ \int_{\mathcal{X}} e^{-2i\langle \xi, x-u-y \rangle} \left( \int_{\mathcal{X}^*} e^{-2i\langle \zeta', y \rangle} \chi_{R_4}(\xi - \zeta') \tilde{f}(u, \zeta') d\zeta' \right) \omega^{\tau_z B}(x, y, x-u) \chi_{R_1}(y) dy \right] du \Big\} d\zeta \equiv \\ & \equiv \int_{\mathcal{X}^*} e^{2i\langle \zeta, x \rangle} \langle \zeta \rangle^m \chi_{R_3}(\xi - \zeta) \left( \int_{\mathcal{X}} e^{-2i\langle \zeta, u \rangle} \Theta_{(R_j, z)}(x, \xi, u) du \right) d\zeta. \end{aligned} \quad (2.64)$$

Let us study closer the continuous function introduced in (2.64):

$$\begin{aligned} \Theta_{(R_j, z)}(x, \xi, u) &:= \tilde{\psi}_{s,r}(u) \chi_{R_2}(x-u) \times \\ & \times \left[ \int_{\mathcal{X}} e^{-2i\langle \xi, x-u-y \rangle} \left( \int_{\mathcal{X}^*} e^{-2i\langle \zeta', y \rangle} \chi_{R_4}(\xi - \zeta') \tilde{f}(u, \zeta') d\zeta' \right) \omega^{\tau_z B}(x, y, x-u) \chi_{R_1}(y) dy \right]; \end{aligned} \quad (2.65)$$

we make the change of variable  $\mathcal{X} \ni y \mapsto v := x-u-y \in \mathcal{X}$  that allow us to write it as:

$$\Theta_{(R_j, z)}(x, \xi, u) := \tilde{\psi}_{s,r}(u) \chi_{R_2}(x-u) T_{R_1, R_4, z}(x, \xi, u), \quad (2.66)$$

$$\begin{aligned} T_{R_1, R_4, z}(x, \xi, u) &:= \\ &= \left[ \int_{\mathcal{X}} e^{-2i\langle \xi, v \rangle} \left( \int_{\mathcal{X}^*} e^{-2i\langle \zeta', x-u-v \rangle} \chi_{R_4}(\xi - \zeta') \tilde{f}(u, \zeta') d\zeta' \right) \omega^{\tau_z B}(x, x-u-v, x-u) \chi_{R_1}(x-u-v) dv \right]. \end{aligned} \quad (2.67)$$

We recall that

$$\tilde{f} := (2\pi)^{d/2} (\mathbf{1} \otimes \mathcal{F}_{\mathcal{X}}^-) f \quad (2.68)$$

and the fact that the distribution  $f \in \mathcal{S}'(\mathcal{X} \times \mathcal{X})$  defined in (2.57) is in fact a smooth function of class  $S^0(\mathcal{X})$  in the first variable and of class  $S^{r-t}(\mathcal{X})$  in the second variable uniformly with respect to the first variable and thus, for  $t > r + (d/2) > d$ , it belongs to  $S^0(\mathcal{X}; L^2(\mathcal{X}))$ . Thus, using the Fourier inversion Theorem and noticing that for any  $g \in S^0(\mathcal{X}; L^2(\mathcal{X}))$  we have that  $\|(\mathbf{1} \otimes \tau_{-u})g(u, \cdot)\|_{L^2(\mathcal{X})} = \|g(u, \cdot)\|_{L^2(\mathcal{X})}$ , we conclude that the tempered distributions

$$T_{R_4}(x, \xi, u, v) := \int_{\mathcal{X}^*} e^{-2i\langle \zeta', x-u-v \rangle} \chi_{R_4}(\xi - \zeta') \tilde{f}(u, \zeta') d\zeta', \quad R_4 \in [1, \infty) \quad (2.69)$$

are a family of functions of class  $S^0(\mathcal{X}; L^2(\mathcal{X}))$  with respect to the variables  $(u, v) \in \mathcal{X} \times \mathcal{X}$  and by the Dominated Convergence Theorem

$$\forall (x, \xi) \in \Xi, \quad \exists \lim_{R_4 \nearrow \infty} T_{R_4}(x, \xi, u, v) = (2\pi)^d f(u, 2(x-u-v)), \text{ in } S^0(\mathcal{X}; L^2(\mathcal{X})) \quad (2.70)$$

uniformly with respect to  $(x, \xi) \in \Xi$ . Due to the fact that by definition we have that  $\omega^{\tau_z B} \in BC_u(\mathcal{X}^3)$  uniformly and smoothly for  $z \in \mathcal{X}$  we conclude that

$$\begin{aligned} \forall (z, x, \xi) \in \mathcal{X} \times \Xi, \quad \exists \lim_{R_4 \nearrow \infty} T_{R_4}(x, \xi, u, v) \omega^{\tau_z B}(x, x-u-v, x-u) \chi_{R_1}(x-u-v) = \\ = (2\pi)^d f(u, 2(x-u-v)) \omega^{\tau_z B}(x, x-u-v, x-u) \chi_{R_1}(x-u-v) =: \theta^{\tau_z B}(x, u, v), \end{aligned} \quad (2.71)$$

in  $S^0(\mathcal{X}; L^2(\mathcal{X}))$  uniformly with respect to  $(z, x, \xi) \in \mathcal{X} \times \Xi$ . Moreover, for any magnetic field  $B$  with components of class  $BC^\infty(\mathcal{X})$  we have that for any  $(x, u) \in \mathcal{X}^2$

$$|\theta^B(x, u, v)| \leq C(B) f(u, 2(x-u-v)) = C(B) < 2(x-u-v) >^{r-t} \quad (2.72)$$

and thus

$$\sup_{(x, u) \in \mathcal{X}^2} \|\theta^B(x, u, \cdot)\|_{L^2(\mathcal{X})} \leq C(B) \|q_{r-t}\|_{L^2(\mathcal{X})}, \quad (2.73)$$

with  $C(B)$  depending only on some seminorm of  $BC^\infty(\mathcal{X})$  of the components of the magnetic field. We easily conclude that  $\theta^B \in BC_u(\mathcal{X}^3) \cap BC_u(\mathcal{X}_x \times \mathcal{X}_u; L^2(\mathcal{X}_v))$  and the map  $\mathcal{X} \ni z \mapsto \theta^{\tau_z B} \in BC_u(\mathcal{X}_x \times \mathcal{X}_u; L^2(\mathcal{X}_v))$  is smooth and bounded. Using Plancherel Theorem and the Dominated Convergence Theorem we conclude that

$$\forall z \in \mathcal{X}, \quad \exists \lim_{R_1 \rightarrow \infty} \lim_{R_4 \rightarrow \infty} T_{R_1, R_4, z} =: F_z, \quad \text{in } BC_u(\mathcal{X}_x \times \mathcal{X}_u; L^2(\mathcal{X}_\xi^*)), \quad (2.74)$$

uniformly with respect to  $z \in \mathcal{X}$ . Moreover, taking into account the properties of the function  $f$  defined in (2.57) (see also Proposition 2.4), we may also conclude that  $F_z(x, \cdot, u) \in \mathcal{S}(\mathcal{X}^* \setminus \{0\})$  uniformly with respect to the variable  $z \in \mathcal{X}$ .

Finally, noticing that by Proposition 2.4,  $\tilde{\psi}_{s,r} \in L^2(\mathcal{X})$  for any  $(s, r) \in \mathbb{R}_+ \times \mathbb{R}_+$  we conclude that

$$\forall z \in \mathcal{X}, \quad \exists \lim_{R_1 \rightarrow \infty} \lim_{R_2 \rightarrow \infty} \lim_{R_4 \rightarrow \infty} \Theta_{(R_j, z)} = (1 \otimes 1 \otimes \tilde{\psi}_{s,r}) F_z, \quad \text{in } L^2(\mathcal{X}_u; BC_u(\mathcal{X}_x; L^2(\mathcal{X}_\xi^*))) \quad (2.75)$$

uniformly for  $z \in \mathcal{X}$ .

In order to control the factor  $\langle \zeta \rangle^m$  in the first integral in (2.63), that we consider as a Fourier transform of a tempered distribution, let us study now the derivatives of  $\Theta_{(R_j, z)}$  with respect to the variable  $u \in \mathcal{X}$ :

$$(\partial_u^\alpha \Theta_{(R_j, z)})(x, \xi, u), \quad |\alpha| = p \in \mathbb{N}^*. \quad (2.76)$$

By Proposition 2.4 we know that for  $s > d$  we have that  $\tilde{\psi}_{s,r} \in \mathcal{H}^p(\mathcal{X})$  for  $p < (s/2)$  and thus all the derivatives  $\partial^\alpha \tilde{\psi}_{s,r}$  are of class  $L^2(\mathcal{X})$  for  $s > 2|\alpha|$ . Let us study the behaviour of the distributions

$$\partial_u^\alpha F_z(x, \xi, u), \quad |\alpha| = p \in \mathbb{N}^*. \quad (2.77)$$

When computing  $\partial_u^\alpha T_{R_1, R_4, z}$ , using Leibnitz rule we have to control the derivatives of order up to  $p \in \mathbb{N}$  with respect to  $u \in \mathcal{X}$  of  $f(u, 2(x - u - v))$ , of  $\omega^{\tau_z B}(x, x - u - v, x - u)$  and of the cut-off functions. Now,  $\partial_u^\alpha f(u, 2(x - u - v))$  is easy to compute and it is clearly a function of class  $S^{-|\alpha|}(\mathcal{X})$  with respect to the first variable and of class  $S^{r-t-|\alpha|}(\mathcal{X})$  with respect to the second variable uniformly with respect to the first variable, and thus for any  $p \in \mathbb{N}$  these functions have the same properties as the function  $f$  in (2.57). Using then Lemma 1.1 in [11] we know that we have the estimations

$$(\partial_y^\alpha \partial_{y'}^\beta \omega^{\tau_z B})(x, y, y') = \theta_{\alpha, \beta}^{\tau_z B}(x, y, y') (\langle x \rangle + \langle y \rangle + \langle y' \rangle)^{|\alpha|+|\beta|} \quad (2.78)$$

where  $\theta_{\alpha, \beta}^{\tau_z B} \in BC_u(\mathcal{X}^3)$  uniformly in  $z \in \mathcal{X}$ . In conclusion we can write  $\partial_u^\alpha \omega^{\tau_z B}(x, x - u - v, x - u)$  as a finite sum of terms of the form  $\theta^{\tau_z B}(x, u, v) \langle x \rangle^p \langle u \rangle^p \langle x - u - v \rangle^p$  with  $\theta^{\tau_z B} \in BC_u(\mathcal{X}^3)$  uniformly in  $z \in \mathcal{X}$ . We get rid of the growing factor  $\langle u \rangle^p$  by replacing  $\tilde{\psi}_{s,r}$  by  $\tilde{\psi}_{s, r+p}$  that has the same properties as  $\tilde{\psi}_{s,r}$ . The factor  $\langle x - u - v \rangle^p$  may be absorbed in the factor  $f$  without changing its properties that we used above, as long as  $t > p + r + (d/2)$ . We remain with the factor  $\langle x \rangle^p$ ; in order to control its growth at infinity we turn back at formula (2.63) and notice that

$$\begin{aligned} & \widetilde{\Phi^{(3)}}_{R_j, z}(x, \xi) = \\ & = \int_{\mathcal{X}^*} \left( \frac{(1 - \Delta_\zeta)^{p/2}}{\langle 2x \rangle^p} e^{2i\langle \zeta, x \rangle} \right) \langle \zeta \rangle^m \chi_{R_3}(\xi - \zeta) \left\{ \int_{\mathcal{X}} e^{-2i\langle \zeta, u \rangle} \tilde{\psi}_{s,r}(u) \chi_{R_2}(x - u) \Theta_{(R_j, z)}(x, \xi, u) \right\}. \end{aligned} \quad (2.79)$$

Considering the  $\zeta$ -integral in the sense of distributions we can transfer the differential operator  $(1 - \Delta_\zeta)^{p/2}$  on the  $\mathcal{S}(\mathcal{X}^*)$  function

$$\mathcal{X}^* \ni \zeta \mapsto \langle \zeta \rangle^m \chi_{R_3}(\xi - \zeta) \left\{ \int_{\mathcal{X}} e^{-2i\langle \zeta, u \rangle} \tilde{\psi}_{s,r}(u) \chi_{R_2}(x - u) \Theta_{(R_j, z)}(x, \xi, u) \right\} \in \mathbb{C}. \quad (2.80)$$

Using the well known facts that  $(1 - \Delta)^{-1/2}$  and  $(1 - \Delta)^{-1/2} \partial_j$  are bounded operators in  $L^2(\mathcal{X}^*)$  we notice that for  $p \in \mathbb{N}$  we can write:

$$(1 - \Delta_\zeta)^{p/2} = \sum_{|\alpha| \leq p} X_\alpha \partial_\zeta^\alpha \quad (2.81)$$

with  $X_\alpha \in \mathbb{B}(L^2(\mathcal{X}^*))$  for any  $\alpha \in \mathbb{N}^d$ . Then we only have to notice that  $\partial_\zeta^\alpha \langle \zeta \rangle^m$  is a symbol of type  $S^m(\mathcal{X}^*)$  for any  $\alpha \in \mathbb{N}^d$  and

$$\partial_\zeta^\alpha e^{-2i\langle \zeta, u \rangle} = (-2i)^{|\alpha|} u^\alpha e^{-2i\langle \zeta, u \rangle}$$

and we can control the factor  $u^\alpha$  by  $\langle u \rangle^{|\alpha|}$  that can be absorbed in  $\tilde{\psi}_{s,r}$  for any  $|\alpha| \in \mathbb{N}$  without changing its properties needed for the arguments above to hold. Finally we notice that all the terms containing derivatives of the cut-off functions  $\chi_{R_j}$  clearly go to 0 when  $R_j \rightarrow \infty$  by the Lebesgue Dominated Convergence Theorem. In conclusion, for  $s > 2m$ , all the derivatives  $\partial_u^\alpha \Theta_{(R_j, z)}$  are functions of class  $L^2(\mathcal{X}_u; BC_u(\mathcal{X}_x; L^2(\mathcal{X}_\xi^*)))$  uniformly for  $z \in \mathcal{X}$  and choosing  $m = [d/2] + 1$ , in the first integral in (2.63) considered as a Fourier transform of a tempered distribution, we intertwine the multiplication with  $\langle \zeta \rangle^m$  with the Fourier transform with respect

to the variable  $u \in \mathcal{X}$ . We use formula (2.81) once again and the Plancherel Theorem noticing that for any  $F \in L^2(\mathcal{X}_u; BC_u(\mathcal{X}_x; L^2(\mathcal{X}_\xi^*)))$ , with  $\|F\|^2 := \int_{\mathcal{X}} \sup_{x \in \mathcal{X}} \int_{\mathcal{X}^*} |F(x, \xi, u)|^2 d\xi du$  we have that

$$\int_{\mathcal{X}} \int_{\mathcal{X}^*} |F(x, \xi, x)|^2 d\xi dx \leq \int_{\mathcal{X}} \sup_{y \in \mathcal{X}} \int_{\mathcal{X}^*} |F(y, \xi, x)|^2 d\xi dx = \|F\|^2. \quad (2.82)$$

This proves that our distribution  $\widetilde{\Phi^{(3)}}_{R_j, z}$  is in fact a function of class  $L^2(\Xi)$  uniformly for  $z \in \mathcal{X}$ .

The term with  $\lambda > 0$  replacing  $\langle \xi - \eta \rangle^m$  will also define a function of class  $L^2(\Xi)$  evidently. The uniformity with respect to  $z \in \mathcal{X}$  follows directly from the above remarks concerning the translation invariance of the bounds. Sumarizing we must have:

$$r > d/2, \quad m > d/2, \quad p = m = [d/2] + 1, \quad t > r + p + d/2 > 3d/2, \quad s > 2m = \begin{cases} d + 1, & \text{if } d = 2p \\ d + 2, & \text{if } d = 2p + 1. \end{cases} \quad (2.83)$$

□

Putting now together Corollary 2.7 and Proposition 2.8 and noticing that  $t(d) > 3d/2$  we obtain the following result.

**Theorem 2.9.** *Suppose given a magnetic field  $B$  with components of class  $BC^\infty(\mathcal{X})$  and suppose fixed some vector potential  $A$  for  $B$  with components of class  $C_{\text{pol}}^\infty(\mathcal{X})$ . If  $s \geq s(d)$ ,  $t \geq t(d)$  and  $f \in \mathcal{S}'(\Xi)$ , then:*

1. *If  $\mathfrak{L}_{s,t}f \in L^\infty(\Xi)$ , then  $\mathfrak{Op}^A(f) \in \mathbb{B}(\mathcal{H})$  and  $\|\mathfrak{Op}^A(f)\|_{\mathbb{B}(\mathcal{H})} \leq C\|\mathfrak{L}_{s,t}f\|_{L^\infty(\Xi)}$ .*
2. *If  $\mathfrak{L}_{s,t}f \in L^p(\Xi)$  for some  $p \in [1, \infty)$ , then  $\mathfrak{Op}^A(f) \in \mathbb{B}_p(\mathcal{H})$  and  $\|\mathfrak{Op}^A(f)\|_{\mathbb{B}_p(\mathcal{H})} \leq C\|\mathfrak{L}_{s,t}f\|_{L^p(\Xi)}$ .*
3. *If  $\mathfrak{L}_{s,t}f \in L^\infty(\Xi)$  and  $\lim_{(x,\xi) \rightarrow \infty} (\mathfrak{L}_{s,t}f)(x, \xi) = 0$  then  $\mathfrak{Op}^A(f) \in \mathbb{B}_\infty(\mathcal{H})$ .*

This result evidently implies Theorem 1.2 and moreover results similar to those obtained in Section 6 from [1].

### 3 Appendix

In this Appendix we prove a simplified version of Theorem 2.2 in [11], that is enough for our analysis in this paper. Not only that in this special case the proof is much simpler then the one in [11] but we also put into evidence the dependence on the magnetic field.

**Proposition 3.10.** *For a magnetic field  $B$  with components of class  $BC^\infty(\mathcal{X})$  the 'magnetic' Moyal product*

$$S_1^m(\mathcal{X}) \times S_1^p(\mathcal{X}) \ni (f, g) \mapsto f \sharp^B g \in S_1^{m+p}(\mathcal{X})$$

*is continuous for the Fréchet topologies being equicontinuous for  $B_{jk} \in \mathcal{B}, \forall (j, k)$ , with  $\mathcal{B} \subset BC^\infty(\mathcal{X})$  any bounded subset for its Fréchet topology.*

*Proof.* Via a standard cut-off procedure it is enough to consider  $(f, g) \in \mathcal{S}(\Xi) \times \mathcal{S}(\Xi)$  and to prove that there exist some finite constants  $C_{M,N} > 0$  such that

$$\nu_{M,N}^{m+p-N}(f \sharp^B g) \leq C_{M,N} \nu_{M-1}(B) \nu_{m_1, n_1}^{m-n_1}(f) \nu_{m_2, n_2}^{p-n_2}(g) \quad (3.84)$$

where we have considered the seminorms indexed by  $n \in \mathbb{N}$ :

$$\nu_n(F) := \sup_{x \in \mathcal{X}} \sup_{|\alpha| \leq n} |(\partial_x^\alpha F)(x)| \quad (3.85)$$

defining the Fréchet topology on  $BC^\infty(\mathcal{X})$  and

$$\nu_n(B) := \max_{j,k} \nu_n(B_{jk}). \quad (3.86)$$

Thus let us compute

$$\begin{aligned} & \langle \xi \rangle^{-(m+p-|\beta|)} \left( \partial_x^\alpha \partial_\xi^\beta (f \sharp^B g) \right) (x, \xi) := \\ & \pi^{-2d} \langle \xi \rangle^{-(m+p-|\beta|)} \partial_x^\alpha \partial_\xi^\beta \left( \int_{\Xi \times \Xi} e^{-2i\sigma(Y, Z)} \omega^B(x, y, z) f(x - y, \xi - \eta) g(x - z, \xi - \zeta) dY dZ \right), \end{aligned} \quad (3.87)$$

that is a finite linear combination of terms of the form

$$\int_{\Xi \times \Xi} e^{-2i\sigma(Y,Z)} (\partial_x^{\alpha_3} \omega^B(x, y, z)) \langle \xi \rangle^{-(m-|\beta_1|)} (\partial_x^{\alpha_1} \partial_\xi^{\beta_1} f)(x-y, \xi-\eta) \langle \xi \rangle^{-(p-|\beta_2|)} (\partial_x^{\alpha_2} \partial_\xi^{\beta_2} g)(x-z, \xi-\zeta) dY dZ$$

with  $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$  and  $\beta_1 + \beta_2 = \beta$ .

In order to estimate these integrals we insert the integrable factor  $\langle y \rangle^{-2n} \langle z \rangle^{-2n} \langle \eta \rangle^{-2n} \langle \zeta \rangle^{-2n}$  with  $(d/2) < n \in \mathbb{N}$  and get rid of the growing factors by the usual integration by parts trick using the identities

$$\partial_{y_j} e^{-2i\sigma(Y,Z)} = 2i\zeta_j e^{-2i\sigma(Y,Z)}, \quad \partial_{z_j} e^{-2i\sigma(Y,Z)} = -2i\eta_j e^{-2i\sigma(Y,Z)}, \quad (3.88)$$

$$\partial_{\eta_j} e^{-2i\sigma(Y,Z)} = -2iz_j e^{-2i\sigma(Y,Z)}, \quad \partial_{\zeta_j} e^{-2i\sigma(Y,Z)} = 2iy_j e^{-2i\sigma(Y,Z)}. \quad (3.89)$$

□

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